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An optimal replenishment policy for deteriorating items with effective investment in preservation technology

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ABSTRACT

In this paper, considering the amount invested in preservation technology and the replenishment schedule as decision variables, we formulate an inventory model with a time-varying rate of deterioration and partial backlogging. The objective is to find the optimal replenishment and preservation technology investment strategies while maximizing the total profit per unit time. For any given preservation technology cost, we first prove that the optimal replenishment schedule not only exists but is unique. Next, under given replenishment schedule, we show that the total profit per unit time is a concave function of preservation technology cost. We then provide a simple algorithm to figure out the optimal preservation technology cost and replenishment schedule for the proposed model. We use numerical examples to illustrate the model.

1. Introduction

In the literature of inventory theory, deteriorating inventory models have been continually modified so as to accommodate more practical features of the real inventory systems. In general, the phenomena of deterioration are vaporization, damage, spoilage, dryness and so on. Certain products such as medicine, volatile liquids, blood, food and many others deteriorate with time resulting in a decreasing usefulness from the original one. The analysis of deteriorating inventory began with Ghare and Schrader (1963), who established the classical no-shortage inventory model with a constant rate of decay. However, it has been empirically observed that failure and life expectancy of many items can be better expressed by a variable deterioration rate. Covert and Philip (1973) extended Ghare and Schrader's model to allow for a variable deterioration rate with a two-parameter Weibull distribution. When determining the optimal inventory policy for that type of products, we cannot ignore the loss due to deterioration. As a result, researchers including Philip (1974), Misra (1975), Tadikamalla (1978), Wee (1997), Chakrabarty et al. (1998), Abad (2001), Mukhopadhyay et al. (2004) and Hung (2011) developed economic order quantity models treating the deterioration rate as a time varying function.

It is well known that certain products such as refrigerated food, medicine, semiconductor chips and many others have a high deterioration rate. Therefore, many enterprises have studied the causes of deterioration and developed preservation technologies to control it and increase the profit. However, the deterioration rate of goods mentioned above is viewed as an exogenous variable, which is not subject to control. In practice, the deterioration rate of products can be controlled and reduced through various efforts like procedural changes and specialized equipment acquisition. The results of the sensitivity analysis in numerous studies (Taso and Sheen, 2008; Yang et al., 2009; Geetha and Uthayakumar, 2010) also showed that a lower deterioration rate is considered beneficial from an economic viewpoint. More recently, to best describe the practical inventory situation, Hsu et al. (2010) proposed a deteriorating inventory with a constant deterioration rate and time-dependent partial backlogging. The main objective in their paper is to find the retailer's replenishment and preservation technology investment strategies which maximize the retailer's unit time profit. The graphical analysis approach is used to show the concavity of the objective function. However, the properties of the retailer's unit time profit had remained unexplored. Furthermore, the preservation technology cost was assumed to be a fixed cost, which is independent of the length of the replenishment cycle. This assumption is unrealistic. For instance, if new equipment, such as refrigeration or temperature controlling equipment, is acquired, the capital cost will be often incorporated into models using an equivalent cost per period or a leasing fee. Procedural changes might involve something like treating individual units of the product, such as by wrapping them in protective material or sealing them in air-tight containers to prevent oxidation. This would involve a cost per unit, in addition to any periodic capital cost. Hence, a more realistic representation of preservation technology

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cost should be a function of the length of replenishment cycle or the order size or both.

The inventory system for deteriorating items has been an object of study for a long time, but little is known about the effect of investing in reducing the rate of product deterioration. In this paper, to obtain robust and general results, we attempt to develop an inventory model for deteriorating items with a generalized deterioration rate and productivity of invested capital. The preservation technology cost is assumed to be a function of the length of replenishment cycle. In addition, we add both time-dependent backorder cost and the cost of lost sales into the total profit. In the next section, the assumptions and notations related to this study are presented. Then, we prove that the optimal replenishment policy not only exists but is unique, with any given preservation technology cost. Next, we show that the total profit per unit time is a concave function of preservation technology cost with a given replenishment schedule. Finally, numerical examples are presented to illustrate the model and the sensitivity analysis in the optimal solutions with respect to parameters of the system is also carried out, which is followed by concluding remarks.

2. Model notations and assumptions

The mathematical model in this paper is developed on the basis of the following notation and assumptions.

2.1. Notations

D = the demand rate per unit time
K = the replenishment cost per order
c = the purchasing cost per unit
p = the selling price per unit, where $p > c$
h = the holding cost per unit time
s = the backorder cost per unit time
π = the goodwill cost of lost sales per unit
ξ = the preservation technology cost per unit time for reducing the deterioration rate in order to preserve the products, $0 \leqslant \xi \leqslant w$, where w is the maximum cost of investment in preservation technology
t_1 = the time at which the inventory level reaches zero
t_2 = the length of period during which shortages are allowed
Q = the ordering quantity per cycle
L = the amount of lost sales per cycle
$I(t)$ = the level of positive inventory at time t
$m(\xi)$ = the proportion of reduced deterioration rate, $0 \leqslant m(\xi) \leqslant 1$
$\Pi(t_1, t_2, \xi)$ = the total profit per unit time

2.2. Assumptions

1. Replenishment rate is infinite, and the lead time is zero.
2. The time horizon of the inventory system is infinite.
3. The proportion of reduced deterioration rate, $m(\xi)$, is a continuous, concave, increasing function of retailer's capital investment. Note that $m''(\xi) < 0$ implies the diminishing marginal productivity of capital.
4. The items deteriorate at a time-varying rate of deterioration $\theta(t)$, where $0 < \theta(t) \ll 1$. There is no repair or replacement of deteriorated units during the replenishment cycle.
5. Shortages are allowed. Unsatisfied demand is backlogged, and the fraction of shortages backordered is $\frac{1}{1+\delta x}$, where x is the waiting time up to the next replenishment and δ is a positive constant. Therefore, if customers do not need to wait, then no sales are lost, and all sales are lost if customers are faced with an infinite wait.

3. Model formulation

Given the assumptions mentioned before, the inventory level follows the pattern depicted in Fig. 1. To establish the total relevant profit function, we consider the following time intervals separately, $[0, t_1)$ and $[t_1, t_1 + t_2)$. During the interval $[0, t_1)$, the inventory is depleted due to the combined effects of demand and deterioration. Hence, the inventory level is governed by the following differential equation:

$$\frac{dI(t)}{dt} = -D - \theta(t)[1 - m(\xi)]I(t), \quad 0 < t < t_1, \tag{1}$$

with the boundary condition $I(t_1) = 0$. Solving the differential equation, we get the inventory level as follows:

$$I(t) = De^{-[1-m(\xi)]g(t)} \int_t^{t_1} e^{[1-m(\xi)]g(u)} du, \quad 0 \leqslant t < t_1, \tag{2}$$

where $g(z) = \int_0^z \theta(u)du$. From (2), we can obtain the number of period-units of inventory carried during the cycle as

$$\int_0^{t_1} I(t)dt = D \int_0^{t_1} e^{-[1-m(\xi)]g(t)} \int_t^{t_1} e^{[1-m(\xi)]g(u)} du dt.$$

At time t_1, the inventory level reaches zero and shortage occurs. During the stockout period, some customers may be willing to wait for a shipping delay while others will leave for another seller because of urgent need. For a customer who desires to purchase the goods at time $t \in [t_1, t_1 + t_2)$, $(t_1 + t_2 - t)$ represents the waiting time up to the next replenishment. Hence, over the time interval $[t_1, t_1 + t_2)$, the inventory level depends only on demand, some of which is lost while a fraction $\frac{1}{1+\delta(t_1+t_2-t)}$ of the demand is backlogged, where $t \in [t_1, t_1 + t_2)$. The inventory level is governed by the following differential equation:

$$\frac{dI(t)}{dt} = -\frac{D}{1 + \delta(t_1 + t_2 - t)}, \quad t_1 < t < t_1 + t_2, \tag{3}$$

with the boundary condition $I(t_1) = 0$. Solving the differential equation (3), we obtain the inventory level as

$$I(t) = -\frac{D}{\delta}\{\ln(1 + \delta t_2) - \ln[1 + \delta(t_1 + t_2 - t)]\}, \quad t_1 \leqslant t < t_1 + t_2. \tag{4}$$

From (4), since $I(t)$ is negative for $t \in [t_1, t_1 + t_2)$, the time-weighted backorders due to shortages during t_1 and $t_1 + t_2$ are

$$B = -\int_{t_1}^{t_1+t_2} I(t)dt = \frac{D[\delta t_2 - \ln(1 + \delta t_2)]}{\delta^2}.$$

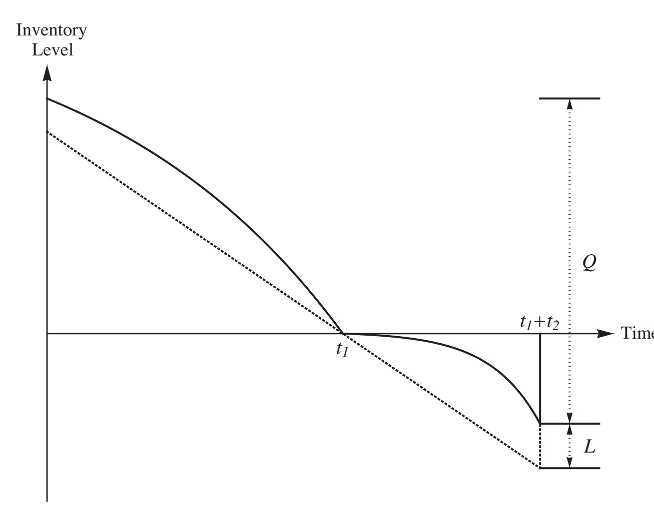


Fig. 1. Graphical representation of the inventory system.

Moreover, the amount of lost sales incurred between time t_1 and $t_1 + t_2$ is

$$L = \int_{t_1}^{t_1+t_2} D\left\{1 - \frac{1}{1+\delta(t_1+t_2-t)}\right\}dt = \frac{D[\delta t_2 - \ln(1+\delta t_2)]}{\delta}.$$

Therefore, the ordering quantity over the replenishment cycle can be determined as

$$\begin{aligned} Q &= \text{initial inventory on hand} + \text{number of backorders} \\ &= I(0) - I(t_1+t_2) = D\int_0^{t_1} e^{[1-m(\xi)]g(u)}du + \frac{D}{\delta}\ln(1+\delta t_2) \end{aligned} \quad (5)$$

and the number of units sold in the replenishment cycle is

$$\int_0^{t_1} Ddt + \int_{t_1}^{t_1+t_2} \frac{D}{1+\delta(t_1+t_2-t)}dt = Dt_1 + \frac{D}{\delta}\ln(1+\delta t_2).$$

We assume that the preservation technology cost depends on the cycle length. During the inventory cycle of time-span $[0, t_1 + t_2)$, the preservation technology cost is $(t_1 + t_2)\xi$. Then, the total profit per unit time of the inventory system can be written as follows:

$$\begin{aligned} \Pi(t_1,t_2,\xi) &= \frac{1}{t_1+t_2}\left\{\begin{matrix} \text{sales revenue} - \text{purchase cost} - \text{ordering cost} \\ -\text{preservation technology cost} - \text{holding cost} \\ -\text{backorder cost} - \text{cost of lost sales} \end{matrix}\right\} \\ &= (p-c)D + \frac{1}{t_1+t_2}\{-K - (t_1+t_2)\xi \\ &\quad - \int_0^{t_1}[e^{[1-m(\xi)]g(t)} - 1]dt - hD\int_0^{t_1} e^{-[1-m(\xi)]g(t)} \\ &\quad \times \int_t^{t_1} e^{[1-m(\xi)]g(u)}dudt - \frac{[s+\delta(p-c+\pi)]D}{\delta^2}[\delta t_2 - \ln(1+\delta t_2)]\right\}. \end{aligned} \quad (6)$$

The problem is to determine t_1, t_2 and ξ which maximize $\Pi(t_1,t_2,\xi)$. If $0 < t_1^* < \infty,\ 0 < t_2^* < \infty$ and $0 < \xi^* < \infty$, by Leibniz's rule for integrals, then $t_1^*,\ t_2^*$ and ξ^* must satisfy

$$\begin{aligned} \frac{\partial \Pi(t_1,t_2,\xi)}{\partial t_1} &= \frac{1}{t_1+t_2}\{-cD[e^{[1-m(\xi)]g(t_1)} - 1] - hD \\ &\quad \times \int_0^{t_1} e^{-[1-m(\xi)]g(t)}\left(\frac{\partial}{\partial t_1}\int_t^{t_1} e^{[1-m(\xi)]g(u)}du\right)dt\right\} \\ &\quad - \frac{1}{(t_1+t_2)^2}\left\{-K - cD\int_0^{t_1}[e^{[1-m(\xi)]g(t)} - 1]dt - hD \right. \\ &\quad \times \int_0^{t_1} e^{-[1-m(\xi)]g(t)}\int_t^{t_1} e^{[1-m(\xi)]g(u)}dudt - \frac{[s+\delta(p-c+\pi)]D}{\delta^2} \\ &\quad \times [\delta t_2 - \ln(1+\delta t_2)]\} = \frac{1}{t_1+t_2}\{-cD[e^{[1-m(\xi)]g(t_1)} - 1] \\ &\quad - hD\int_0^{t_1} e^{[1-m(\xi)][g(t_1)-g(t)]}dt\right\} - \frac{1}{(t_1+t_2)^2}\{-K - cD \\ &\quad \times \int_0^{t_1}[e^{[1-m(\xi)]g(t)} - 1]dt - hD\int_0^{t_1} e^{-[1-m(\xi)]g(t)} \\ &\quad \times \int_t^{t_1} e^{[1-m(\xi)]g(u)}dudt - \frac{[s+\delta(p-c+\pi)]D}{\delta^2} \\ &\quad \times [\delta t_2 - \ln(1+\delta t_2)]\} = 0, \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{\partial \Pi(t_1,t_2,\xi)}{\partial t_2} &= -\frac{1}{t_1+t_2}\frac{[s+\delta(p-c+\pi)]D}{\delta}\left(\frac{\delta t_2}{1+\delta t_2}\right) \\ &\quad - \frac{1}{(t_1+t_2)^2}\left\{-K - cD\int_0^{t_1}[e^{[1-m(\xi)]g(t)} - 1]dt \right. \\ &\quad - hD\int_0^{t_1} e^{-[1-m(\xi)]g(t)}\int_t^{t_1} e^{[1-m(\xi)]g(u)}dudt \\ &\quad \left. - \frac{[s+\delta(p-c+\pi)]D}{\delta^2}[\delta t_2 - \ln(1+\delta t_2)]\right\} = 0 \end{aligned} \quad (8)$$

and

$$\begin{aligned} \frac{\partial \Pi(t_1,t_2,\xi)}{\partial \xi} &= -1 + \frac{1}{t_1+t_2}\left\{cD\int_0^{t_1} m'(\xi)g(t)e^{[1-m(\xi)]g(t)}dt \right. \\ &\quad - hD\int_0^{t_1}\left[m'(\xi)g(t)e^{-[1-m(\xi)]g(t)}\int_t^{t_1} e^{[1-m(\xi)]g(u)}du \right. \\ &\quad \left.\left. + e^{-[1-m(\xi)]g(t)}\int_t^{t_1} -m'(\xi)g(u)e^{[1-m(\xi)]g(u)}du\right]dt\right\} \\ &= -1 + \frac{1}{t_1+t_2}\left\{cD\int_0^{t_1} m'(\xi)g(t)e^{[1-m(\xi)]g(t)}dt \right. \\ &\quad \left. - hD\int_0^{t_1}\int_t^{t_1} m'(\xi)[g(t)-g(u)]e^{-[1-m(\xi)][g(t)-g(u)]}dudt\right\} = 0. \end{aligned} \quad (9)$$

After some algebraic manipulation, the equivalents of (7) and (8) then become

$$\frac{s+\delta(p-c+\pi)}{\delta}\frac{\delta t_2}{1+\delta t_2} = c[e^{[1-m(\xi)]g(t_1)} - 1] + h \times \int_0^{t_1} e^{[1-m(\xi)][g(t_1)-g(t)]}dt \quad (10)$$

and

$$\begin{aligned} &-K - cD\int_0^{t_1}[e^{[1-m(\xi)]g(t)} - 1]dt - hD\int_0^{t_1} e^{-[1-m(\xi)]g(t)} \\ &\quad \times \int_t^{t_1} e^{[1-m(\xi)]g(u)}dudt \\ &\quad - \frac{[s+\delta(p-c+\pi)]D}{\delta^2}\left[\delta t_2 - \ln(1+\delta t_2) - \frac{\delta^2 t_2(t_1+t_2)}{1+\delta t_2}\right] \\ &\quad = 0. \end{aligned} \quad (11)$$

Applying (10) and (11), we obtain the following results.

Proposition 1. *For any given ξ, we have*

(a) *The system of (7) and (8) has a unique solution.*
(b) *The solution in (a) satisfies the second order conditions for a global maximum of $\Pi(t_1,t_2|\xi)$.*

Proof. Please see Appendix A for details.

With the help of Proposition 1 we know that, for any given ξ, the point (t_1^*, t_2^*) which maximizes the retailer's unit time profit not only exists but is unique. Because a closed-form solution to (7) and (8) cannot be found directly, an alternative is to use some iterative method. From (10), it is trivial to see that t_2 can be expressed as a function of t_1 and ξ, and this result reduces the profit function from a 3-dimensional space (t_1,t_2,ξ) to a 2-dimensional space (t_1,ξ). For any given ξ, substituting (A.1) into (A.3), we can obtain the value of t_1^* from $G(t_1) = 0$ by using Newton Method (or its modifications). Then, t_2^* follows immediately from (A.1).

When $\Pi(t_1,t_2|\xi)$ is solved for a given ξ, $\Pi(t_1,t_2,\xi)$ should improve for fixed values of t_1 and t_2. Next, we study the conditions under which the optimal preservation technology cost not only exists but is unique. For any given feasible t_1 and t_2, taking the second partial derivative of (6) with respect to ξ yields

$$\begin{aligned} \frac{\partial^2 \Pi(\xi|t_1,t_2)}{\partial \xi^2} &= \frac{1}{t_1+t_2}\left\{cD\int_0^{t_1}\{m''(\xi)g(t)e^{[1-m(\xi)]g(t)} \right. \\ &\quad - [m'(\xi)g(t)]^2 e^{[1-m(\xi)]g(t)}\}dt \\ &\quad \left. - hD\int_0^{t_1}\int_t^{t_1}\{\cdot\}e^{-[1-m(\xi)][g(t)-g(u)]}dudt\right\}, \end{aligned} \quad (12)$$

where

$\{\cdot\} = \{m''(\xi)[g(t) - g(u)] + \{m'(\xi)[g(t) - g(u)]\}^2\}.$

The following proposition gives a result concerning the uniqueness of optimal solution for the total profit per unit time $\Pi(\xi|t_1,t_2)$. □

Proposition 2. *For any given feasible (t_1,t_2), if the productivity of invested capital, $m(\xi)$, is a strictly concave function of ξ (i.e. $m''(\xi) < 0$ or diminishing marginal productivity of capital), then there exists a unique preservation technology cost ξ^* such that $\Pi(\xi|t_1,t_2)$ is maximum.*

Proof. Please see Appendix B for details.

Thus at present, we can realize that, for any given feasible (t_1,t_2), there exists a unique ξ such that $\Pi(\xi|t_1,t_2)$ is maximum. Because ξ is bounded over $[0,w]$, the above derivation also indicates that the optimal ξ^* should be selected to satisfy

$$\frac{\partial \Pi(\xi|t_1,t_2)}{\partial \xi} = 0, \quad \text{otherwise } \xi^* = \begin{cases} 0, & \text{if } \left.\frac{\partial \Pi(\xi|t_1,t_2)}{\partial \xi}\right|_{\xi=0} < 0, \\ w, & \text{if } \left.\frac{\partial \Pi(\xi|t_1,t_2)}{\partial \xi}\right|_{\xi=w} > 0. \end{cases} \tag{13}$$

Combining Propositions 1 and 2, we propose the following algorithm to determine the solution for the proposed model. □

Algorithm 1.

1. Start with $j = 0$ and the initial trial value of ξ_j, where ξ_j is the root of $m(\xi) = 0.5$.
2. Find the optimal t_1 and t_2 from (A.1) and (A.3), for a given preservation technology cost ξ_j.
3. Use the result gained from Step 2, and then determine the optimal ξ_{j+1} by (13).
4. If the difference between ξ_j and ξ_{j+1} is sufficiently small, set $\xi^* = \xi_{j+1}$, then (t_1^*, t_2^*, ξ^*) is the optimal solution and stop. Otherwise, set $j = j + 1$ and return to Step 2.

To begin the search, we need a starting value for ξ. Note that since $m(\xi)$ is concave and the value of $m(\xi)$ is between 0 and 1, we might choose $m(\xi_0) = 0.5$ for our initial guess. Let the root of $m(\xi) = 0.5$ be the initial trial value of ξ in Step 1. In Step 2, where Proposition 1 is applied, there should be a unique local maximum solution (t_1^*, t_2^*) of $\Pi(t_1,t_2,\xi)$ for the current ξ. Then the value of $\Pi(t_1,t_2,\xi)$ is improving in Step 3 where Proposition 2 is applied. Thus, the procedure repeating Steps 2 and 3 would converge to a local maximum of $\Pi(t_1,t_2,\xi)$, which will be the global maximum. By using several starting values of ξ, the algorithm can be repeated to identify the global maximum solution.

4. Numerical example

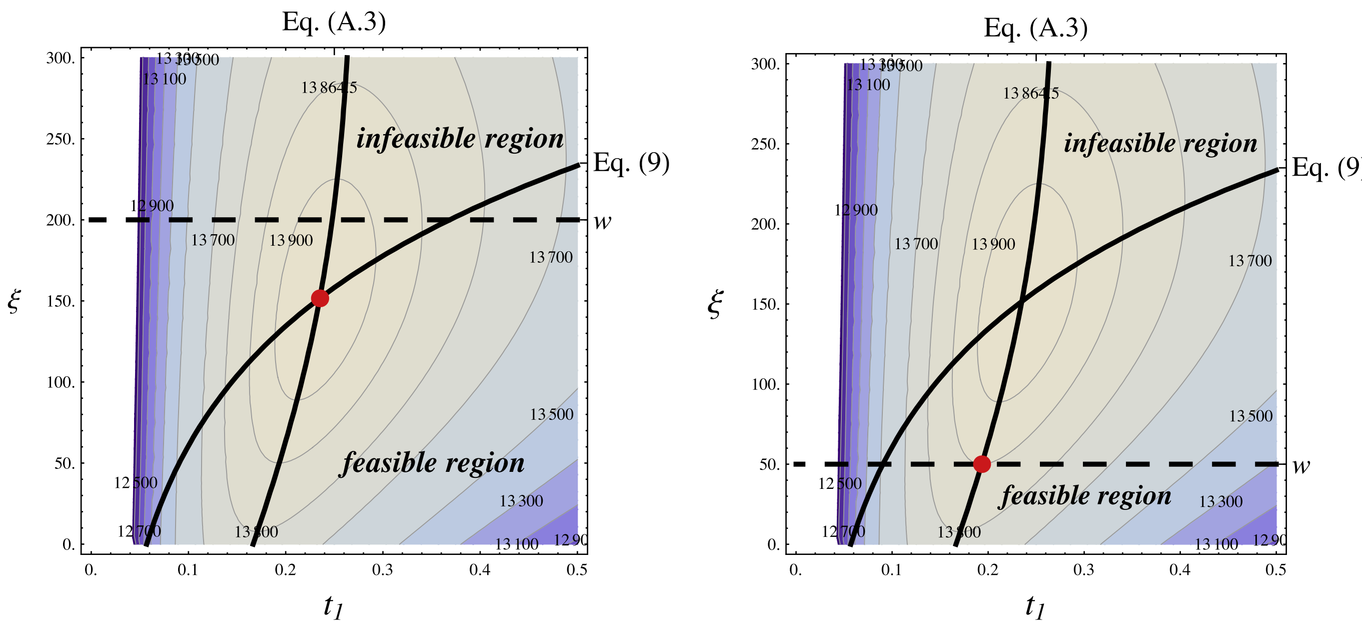
Example 1. In order to illustrate the above solution procedure, we consider an inventory situation where $K = 120$per order, $p = 35$/per unit, $c = 20$/per unit, $h = 3$/per unit/per year, $s = 4$/per unit/per year, $\pi = 5$/per unit, $D = 1000$/per year, $\theta(t) = 0.2 + 0.1t$, $\delta = 2$. The reduced deterioration rate is $m(\xi) = 1 - e^{-a\xi}$, $a \geqslant 0$. We set $a = 0.01$ and the constraint of the preservation technology cost $w = 200$. By solving $m(\xi) = 0.5$, we obtain $\xi_0 = 69.3147$. Then, by applying the Algorithm 1, we have $\xi^* = 151.5916$, $t_1^* = 0.2351$, $t_2^* = 0.0220$, $\Pi^* = 13919.3$ and $Q^* = 257.9$. To show the iterative scheme obtained by using Algorithm 1, the contour plot of objective and Eqs. (9) and (A.3) is depicted in Fig. 2(a). From Fig. 2(a), it is easy to see that the objective is concave and the solution will converge to a unique point in a finite number of iterations. Meanwhile, the three-dimensional retailer's unit time profit graph for $\xi^* = 151.5916$ is shown in Fig. 3.

Next, we run the numerical results with distinct values of $\xi = 0,20,40,\ldots,300$. For the 16 given values of ξ, we use Proposition 1 to find their corresponding optimal values of t_1^*, t_2^* and $\Pi(t_1^*,t_2^*|\xi)$ respectively. The numerical results are shown in Table 1. Then we take the 16 values of $(\xi, \Pi(t_1^*,t_2^*|\xi))$ to depict Fig. 4. From Fig. 4, it can be seen that $\Pi(\xi|t_1^*,t_2^*)$ is concave under the 16 given values of (ξ,t_1^*,t_2^*). Furthermore, the numerical result of Table 1 shows that increasing the preservation technology investment results in an increase in the service level $(t_1^*/(t_1^*+t_2^*))$, while the improvement involves a tradeoff between the investment of preservation technology and the total profit per unit time.

Example 2. In this example, the same data as in Example 1 are used except putting limited capital $w = 50$. From Example 1, we know that $\Pi(t_1,t_2,\xi)$ reaches its maximum at $\xi = 151.5916$. Because $\Pi(t_1,t_2,\xi)$ is a strictly concave function of ξ, it follows that $\xi^* = w = 50$. Then, by Proposition 1, we get $t_1^* = 0.1934$ and $t_2^* = 0.0259$. The retailer's unit time profit obtained here is $\Pi^* = 13864.5$ and the contour plot of objective and Eqs. (9) and (A.3) is depicted in Fig. 2(b).

Next, we study the sensitivity of the optimal solution by changing the values of the different parameters associated with the model. Applying the algorithm's procedures yields the results reported in Table 2. The results obtained for illustrative examples provide certain insights about the problem studies. Some of them are as follows.

1. When the replenishment cost per order (K) increases, t_1^*, t_2^*, ξ^*, Q^* and the service level will increase; however, Π^* will decrease. This implies that when the replenishment cost per order is high, the total profit per unit time is low. In addition, a larger K leads to a longer inventory cycle, which leads to more loss due to deterioration. Then the retailer would like to spend more money on reducing deterioration, i.e., a larger value for ξ^*. In contrast, if the parameter K is low enough, then ξ^* will reach zero such that the inventory system reduces to basic EOQ with partial backordering model without investment in preservation technology.
2. When the purchasing cost per unit (c) increases, t_2^*, ξ^* and Q^* will increase; however, t_1^*, Π^* and the service level will decrease. From economic point of view, if the supplier provides a higher purchasing cost per unit, the retailer will order larger quantity in order to take the benefits of the decreases of the deteriorating items by increasing the preservation technology cost.
3. When the holding cost per unit time (h) increases, t_1^*, ξ^*, Π^*, Q^* and the service level will decrease. It implies that when the holding cost increases, the retailer should decrease the optimal preservation technology cost to avoid too much inventory on hand. However, if h is larger, then t_2^* will increase because backordering has become relatively less expensive than holding inventory.
4. When the parameter a increases, t_1^*, Π^*, Q^* and the service level will increase; however, t_2^* will decreases. From managerial point of view, as a increases, it implies that the marginal return of capital increases. Hence, the optimal total profit per unit time increases due to a lower preservation technology cost and less deterioration. If a is high enough, then the retailer needs to spend less money on reducing deterioration, i.e., a smaller value for ξ^*. On the other hand, if the parameter a is low enough, then



(a) Example 1 (b) Example 2

Fig. 2. Contours of the total profit per unit time, Eqs. (9) and (A.3). The dashed line represents the boundary of the constraint w.

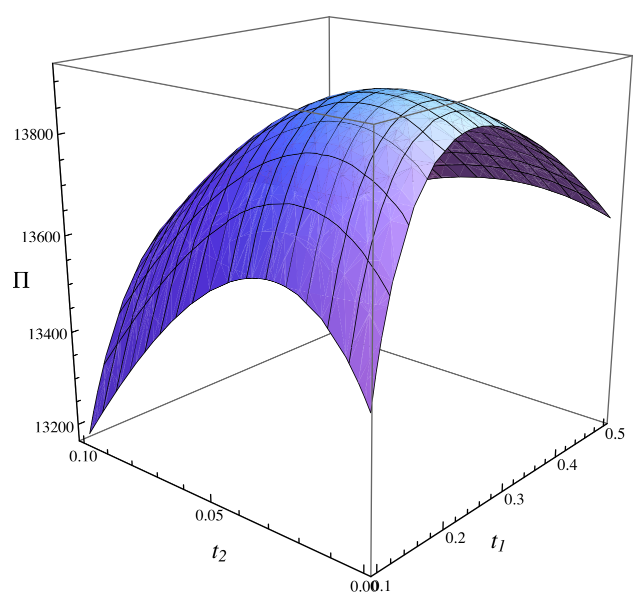


Fig. 3. The retailer's unit time profit, $\Pi(t_1,t_2,151.5916)$.

Table 1
Optimal solution of $\Pi(t_1,t_2|\xi)$ with distinct values of ξ.

ξ	t_1	t_2	$\Pi(t_1^*,t_2^*,\xi^*)$	Service level
0	0.1666	0.0292	13785.0	0.8507
20	0.1777	0.0278	13821.7	0.8647
40	0.1883	0.0265	13851.8	0.8765
60	0.1984	0.0254	13875.7	0.8864
80	0.2078	0.0245	13893.8	0.8947
100	0.2164	0.0236	13906.6	0.9015
120	0.2243	0.0229	13914.7	0.9072
140	0.2314	0.0223	13918.7	0.9119
160	0.2376	0.0218	13919.0	0.9158
180	0.2432	0.0214	13916.1	0.9190
200	0.2479	0.0211	13910.4	0.9217
220	0.2521	0.0208	13902.4	0.9239
240	0.2556	0.0205	13892.4	0.9257
260	0.2586	0.0203	13880.7	0.9271
280	0.2611	0.0202	13867.6	0.9283
300	0.2632	0.0200	13853.3	0.9293

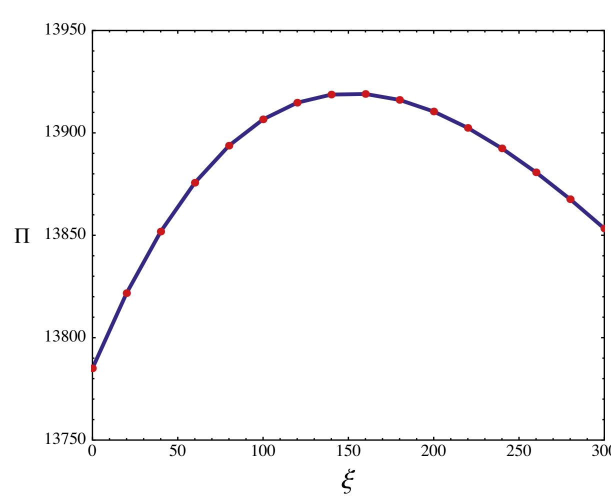


Fig. 4. Graphical representation of $\Pi(\xi|t_1^*,t_2^*)$.

ξ^* will reach zero such that the inventory system reduces to basic EOQ with partial backordering model without that investment.
5. When w is relatively lower, decreasing w decreases t_1^*, ξ^*, Π^*, Q^* and the service level. However, t_2^* increases if w decreases.
6. The total profit per unit time is more sensitive on the change in c. It implies that the effect of c on the total profit per unit time is significant.
7. The parameters, c and h have stronger effect on the optimal service level than the others.

Table 2
Effect of changes in parameters for Example 1.

		−50%	−40%	−30%	−20%	−10%	0	10%	20%	30%	40%	50%
K	t_1^*	0.1562	0.1744	0.1911	0.2066	0.2213	0.2351	0.2483	0.2609	0.2729	0.2846	0.2958
	t_2^*	0.0163	0.0176	0.0188	0.0199	0.0210	0.0220	0.0230	0.0240	0.0249	0.0258	0.0266
	ξ^*	108.1119	119.7837	129.5074	137.8371	145.1209	151.5916	157.4121	162.7011	167.5477	172.0202	176.1725
	Π^*	14199.1	14133.3	14073.5	14018.5	13967.3	13919.3	13873.9	13830.7	13789.5	13750.1	13712.1
	Q^*	173.1	192.6	210.6	227.3	243.0	257.9	272.1	285.7	298.7	311.2	323.3
	Service level	0.9057	0.9084	0.9105	0.9120	0.9132	0.9143	0.9151	0.9158	0.9165	0.9170	0.9175
c	t_1^*	0.2394	0.2388	0.2381	0.2373	0.2363	0.2351	0.2336	0.2318	0.2296	0.2266	0.2226
	t_2^*	0.0151	0.0161	0.0173	0.0186	0.0202	0.0220	0.0243	0.0270	0.0305	0.0350	0.0410
	ξ^*	88.8843	105.8398	120.0324	132.1338	142.5646	151.5916	159.3736	165.9815	171.3956	175.4787	177.9019
	Π^*	23974.4	21958.5	19945.6	17935.0	15926.3	13919.3	11913.9	9910.3	7908.5	5909.0	3912.6
	Q^*	256.7	256.8	256.9	257.1	257.5	257.9	258.5	259.2	260.2	261.3	262.9
	Service level	0.9408	0.9369	0.9324	0.9273	0.9213	0.9143	0.9058	0.8955	0.8827	0.8663	0.8445
h	t_1^*	0.3231	0.2982	0.2780	0.2614	0.2473	0.2351	0.2245	0.2150	0.2066	0.1990	0.1921
	t_2^*	0.0165	0.0177	0.0189	0.0200	0.0211	0.0220	0.0230	0.0238	0.0247	0.0255	0.0262
	ξ^*	188.4109	179.2390	171.2144	164.0491	157.5525	151.5916	146.0698	140.9150	136.0715	131.4957	127.1528
	Π^*	14110.8	14066.6	14025.8	13988.0	13952.6	13919.3	13887.8	13857.9	13829.5	13802.3	13776.3
	Q^*	341.0	317.2	298.1	282.4	269.2	257.9	248.1	239.5	231.8	224.9	218.7
	Service level	0.9516	0.9438	0.9362	0.9288	0.9215	0.9143	0.9072	0.9002	0.8933	0.8865	0.8799
a	t_1^*	0.1999	0.2113	0.2197	0.2260	0.2310	0.2351	0.2385	0.2413	0.2437	0.2457	0.2475
	t_2^*	0.0253	0.0241	0.0234	0.0228	0.0224	0.0220	0.0218	0.0216	0.0214	0.0212	0.0211
	ξ^*	126.2160	146.6500	154.2640	155.8631	154.4796	151.5916	147.9653	144.0150	139.9702	135.9592	132.0531
	Π^*	13815.7	13840.8	13864.1	13884.9	13903.2	13919.3	13933.6	13946.3	13957.7	13967.9	13977.1
	Q^*	225.7	235.9	243.6	249.4	254.1	257.9	261.1	263.8	266.1	268.1	269.9
	Service level	0.8878	0.8975	0.9039	0.9084	0.9117	0.9143	0.9163	0.9179	0.9193	0.9205	0.9214
w	t_1^*	0.2164	0.2243	0.2314	0.2351	0.2351	0.2351	0.2351	0.2351	0.2351	0.2351	0.2351
	t_2^*	0.0236	0.0229	0.0223	0.0220	0.0220	0.0220	0.0220	0.0220	0.0220	0.0220	0.0220
	ξ^*	100.0	120.0	140.0	151.5916	151.5916	151.5916	151.5916	151.5916	151.5916	151.5916	151.5916
	Π^*	13906.6	13914.7	13918.7	13919.3	13919.3	13919.3	13919.3	13919.3	13919.3	13919.3	13919.3
	Q^*	241.3	248.3	254.6	257.9	257.9	257.9	257.9	257.9	257.9	257.9	257.9
	Service level	0.9015	0.9072	0.9119	0.9143	0.9143	0.9143	0.9143	0.9143	0.9143	0.9143	0.9143

5. Concluding remarks

The purpose of this study is to present an inventory model involving controllable deterioration rate to extend the traditional EOQ model. In real markets, the retailer can reduce the deterioration rate of product by making effective capital investment in storehouse equipment. The numerical example results succinctly explained the importance of preservation technology investment. It also indicated that the retailer can reduce the economic losses, and improve customer service level to create more competitive advantages with the investments to reduce the deterioration rate. Furthermore, we can also see that any deterioration rate can be applied to this model such as the three-parameter Weibull deterioration rate (e.g., Philip, 1974) and Gamma deterioration rate (e.g., Tadikamalla, 1978). Hence the utilization of general deterioration rates make the scope of the application broader.

The proposed model can be extended in several ways. For instance, we may consider the permissible delay in payments. Also, we could extend the model with finite replenishment rate. Finally, we could generalize the model to allow for quantity discounts, finite capacity and others.

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Appendix A

A.1. Proof the part (a) of Proposition 1

From (10), after some algebra, we obtain

$$t_2 = \frac{c[e^{[1-m(\xi)]g(t_1)} - 1] + h\int_0^{t_1} e^{[1-m(\xi)][g(t_1)-g(t)]}dt}{[s+\delta(p-c+\pi)] - \delta\left\{c[e^{[1-m(\xi)]g(t_1)} - 1] + h\int_0^{t_1} e^{[1-m(\xi)][g(t_1)-g(t)]}dt\right\}}. \tag{A.1}$$

It is easy to see that, by assumption $t_2 > 0$,

$$\frac{s+\delta(p-c+\pi)}{\delta} > c[e^{[1-m(\xi)]g(t_1)} - 1] + h\int_0^{t_1} e^{[1-m(\xi)][g(t_1)-g(t)]}dt.$$

Since the first derivative of RHS of previous inequality with respect to t_1 is

$$[1-m(\xi)]\theta(t_1)\left\{ce^{[1-m(\xi)]g(t_1)} + h\int_0^{t_1} e^{[1-m(\xi)][g(t_1)-g(t)]}dt\right\} + h > 0$$

and hence, it is a strictly increasing function of t_1. Therefore, the root of

$$\frac{s+\delta(p-c+\pi)}{\delta} = c[e^{[1-m(\xi)]g(t_1)} - 1] + h\int_0^{t_1} e^{[1-m(\xi)][g(t_1)-g(t)]}dt,$$

say $\hat{t}_1$, is the upper bound of t_1.

Next, in order to prove the existence of the solution, differentiating (7) implicitly with respect to t_1 yields

$$\frac{s+\delta(p-c+\pi)}{(1+\delta t_2)^2}\frac{dt_2}{dt_1} = [1-m(\xi)]\theta(t_1)\left\{ce^{[1-m(\xi)]g(t_1)} + h\int_0^{t_1} e^{[1-m(\xi)][g(t_1)-g(t)]}dt\right\} + h > 0 \quad (A.2)$$

and hence $dt_2/dt_1 > 0$. From (11), suppose that

$$G(t_1) = -K - cD\int_0^{t_1}[e^{[1-m(\xi)]g(t)}-1]dt - hD\int_0^{t_1} e^{-[1-m(\xi)]g(t)} \times \int_t^{t_1} e^{[1-m(\xi)]g(u)}dudt - \frac{[s+\delta(p-c+\pi)]D}{\delta^2}\left[\delta t_2 - \ln(1+\delta t_2) - \frac{\delta^2 t_2(t_1+t_2)}{1+\delta t_2}\right]. \quad (A.3)$$

After assembling (10), the implicit differentiation of $G(t_1)$ with respect to t_1 yields

$$\frac{dG(t_1)}{dt_1} = -cD[e^{[1-m(\xi)]g(t_1)}-1] - hD\int_0^{t_1} e^{[1-m(\xi)][g(t_1)-g(t)]}dt + \frac{[s+\delta(s-c+\pi)]D}{\delta}\frac{\delta t_2}{1+\delta t_2} + \frac{[s+\delta(s-c+\pi)]D(t_1+t_2)}{(1+\delta t_2)^2}\frac{dt_2}{dt_1} = \frac{[s+\delta(s-c+\pi)]D(t_1+t_2)}{(1+\delta t_2)^2}\frac{dt_2}{dt_1} > 0.$$

Clearly, $G(t_1)$ is a strictly increasing function of t_1. Moreover, it can be shown that

$$G(0) = -K < 0$$

and

$$\lim_{t\to\hat{t}_1} G(t_1) = -K - cD\int_0^{\hat{t}_1}[e^{[1-m(\xi)]g(t)}-1]dt - hD\int_0^{\hat{t}_1} e^{-[1-m(\xi)]g(t)} \times \int_t^{\hat{t}_1} e^{[1-m(\xi)]g(u)}dudt - \frac{[s+\delta(p-c+\pi)]D}{\delta^2} \times \lim_{t\to\hat{t}_1}\left[\delta t_2 - \ln(1+\delta t_2) - \frac{\delta^2 t_2(t_1+t_2)}{1+\delta t_2}\right] = \infty > 0,\ (\text{by (A.1)}, t_2\to\infty \text{ as } t_1\to\hat{t}_1)$$

thus the Intermediate Value Theorem implies that the root of $G(t_1) = 0$ is unique. This completes the proof of part (a). □

A.2. Proof the part (b) of Proposition 1

Let (t_1^*, t_2^*) be the solution of (7) and (8), then the second-order condition for maximization becomes

$$\left.\frac{\partial^2\Pi(t_1,t_2|\xi)}{\partial t_1^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} = -\frac{D}{t_1^*+t_2^*}\left\{[1-m(\xi)]\theta(t_1^*)\times\left\{ce^{[1-m(\xi)]g(t_1^*)} + h\int_0^{t_1^*} e^{[1-m(\xi)][g(t_1^*)-g(t)]}dt\right\} + h\right\} < 0,$$

$$\left.\frac{\partial^2\Pi(t_1,t_2|\xi)}{\partial t_2^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} = -\frac{1}{t_1^*+t_2^*}\frac{D[s+\delta(p-c+\pi)]}{(1+\delta t_2^*)^2} < 0$$

and

$$\left.\frac{\partial^2\Pi(t_1,t_2|\xi)}{\partial t_1\partial t_2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} = 0.$$

Thus, the determinant of the Hessian matrix at the stationary point (t_1^*, t_2^*) is

$$\det(\mathbf{H}) = \left.\frac{\partial^2\Pi(t_1,t_2|\xi)}{\partial t_1^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} \times \left.\frac{\partial^2\Pi(t_1,t_2|\xi)}{\partial t_2^2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)} - \left[\left.\frac{\partial^2\Pi(t_1,t_2|\xi)}{\partial t_1\partial t_2}\right|_{(t_1,t_2)=(t_1^*,t_2^*)}\right]^2 > 0.$$

Clearly, the Hessian matrix at point (t_1^*, t_2^*) is negative-definite and (t_1^*, t_2^*) represents global maximum point of $\Pi(t_1,t_2|\xi)$. This completes the proof of part (b). □

Appendix B

B.1. Proof of Proposition 2

Proof. Because $t \leqslant u \leqslant t_1$ and $g(z)$ is a strictly increasing function of z, we have $g(t) - g(u) < 0$. Further, since $[m'(\xi)]^2 > 0$ and $m''(\xi) < 0$, we obtain $\{\cdot\} > 0$. Using these results, it follows that $\partial^2\Pi(\xi|t_1,t_2)/\partial\xi^2 < 0$ for any feasible (t_1,t_2). This implies that $\Pi(\xi|t_1,t_2)$ is concave in ξ for given values of t_1 and t_2 and results in a maximum value. □

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